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ON THE STABLE MANIFOLDS OF NONWANDERING SETS KUPKA-SMALE DIFFEOMORPHISMS

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A diffeomorphism f is said to have the Kupka-Smale property if all periodic point of f are hyperbolic and for any periodic points p and q, the stable manifold $W^s(p)$ intersects the unstable manifold $W^u(q)$ transversely.

The purpose of this paper is to give a simple example of a diffeomorphism f with the Kupka-Smale property for which the set of periodic points of f, Per(f) is dense in the nonwandering set of f, $\Omega(f)$ but $\bigcup_{p \in Per(f)} W^s(p)$ is not dense in $W^s(\Omega(f))$.

EXAMPLE

Our example of a diffeomorphism f is defined on the annulas, a product of the interval I = (1, 2) and the circle S^1, by composing three diffeomorphisms ρ, G and H on $I \times S^1$ i.e., $f = \rho \circ G \circ H$.

First we define a diffeomorphism H.

Let $\{t_n\}$ be a sequence of rational numbers in I such that $t_1 < t_2 < \dots < t_n < \dots$ and $t_n \to \sqrt{2}$ as $n \to \infty$. We construct a orientation preserving diffeomorphism h on I as follows :

(1) t_n is a hyperbolic fixed point with $h'(t_n) > 1$ (< 1) for even n (odd n)

(2) $\sqrt{2}$ is a fixed point with $h'(\sqrt{2}) = 1$

(3) $h^m(t) \to \sqrt{2}$ as $m \to \infty$ if $t \in (\sqrt{2}, 2)$

$h^m(t) \to t_n$ as $m \to \infty$ for some odd n if $t \in (1, \sqrt{2}) - \bigcup_n \{t_n\}$.

where $h'(t)$ is the derivative of h at t.

We define a diffeomorphism H by the formula $H(t, \theta) = (h(t), \theta)$ for any $(t, \theta) \in I \times S^1$.

Next we define a diffeomorphism G.

Let g_k be the time one map of a flow ϕ_t on S^1 such that $\phi_t(\theta + 2\pi/k) = \phi_t(\theta) + 2\pi/k$ for any $0 \leqq \theta \leqq 2\pi$ and the flow has 2k-hyperbolic fixed points (see Figure 1). Now we construct a smooth isotopy g : $I \times S^1 \to S^1$ which satisfies the followings :

(4) $g(t_n, \theta) = g_{k(n)}(\theta)$ for any n.

(5) $g(\sqrt{2}, \theta) = \theta$

(6) $\left|\frac{\partial}{\partial t} g(t, \theta)\right| < 2\pi$ for any $(t, \theta) \in I \times S^1$

where $\{k(n)\}$ is a sequence of integers such that $k(1) < \dots < k(n) < \dots$, and $k(n) \cdot t_n$ is an integer for any n.

We define a diffeomorphism G by the formula $G(t, \theta) = (t, g(t, \theta))$ for any $(t, \theta) \in I \times S^1$.

Finally let ρ be a map such that for any $(t, \theta) \in I \times S^1$, $\rho(t, \theta) = (t, \theta + 2\pi \cdot t)$.

LEMMA. <u>A diffeomorphism f has the Kupka-Smale property.</u>

PROOF. First we show $\mathrm{Per}(f)$ is dense in $\Omega(f)$. By the definition of h, $\Omega(f) \subset \bigcup_n \{t_n\} \times S^1 \cup \{\sqrt{2}\} \times S^1$. Since $f \mid \{\sqrt{2}\} \times S^1$ is the irrational ($2\pi\sqrt{2}$) rotation, all elements in $\{\sqrt{2}\} \times S^1$ are nonwandering points but not periodic points. In $\{t_n\} \times S^1$, $f(t_n, \theta) = (t_n, g_{k(n)}(\theta) + 2\pi \cdot t_n)$. Since $g_{k(n)}$ commutes with $2\pi/k(n)$-rotation and $2\pi \cdot t_n = 2\pi N/k(n)$ for some integer N, $f^{k(n)}(t_n, \theta) = (t_n, (g_{k(n)})^{k(n)}(\theta))$. By the definition of $g_{k(n)}$, there are no nonwandering points of f except for $2k(n)$-periodic points of f. Therefore $\Omega(f) = \{\sqrt{2}\} \times S^1 \cup (\bigcup_n (t_n, \theta_{n_j}))$ where θ_{n_j} is a hyperbolic fixed point of $g_{k(n)}$. Since $k(n) \to \infty$ as $n \to \infty$, $|\theta_{n_j} - \theta_{n_{j+1}}| \to 0$. Therefore Per(f) is dense in $\Omega(f)$.

Next we show that periodic points are hyperbolic. From now on, we identify the tangent space of $I \times S^1$ at $x = (t, \theta)$, $T_x(I \times S^1)$ with $T_t(I) \times T_\theta(S^1)$. Then the derivative of f at x, Df(x) is represented by

$$Df(x) = \begin{bmatrix} A(x), & 0 \\ C(x), & B(x) \end{bmatrix}$$

where $A(x) = h'(t)$, $B(x) = \frac{\partial}{\partial\theta}g(x)$ and $C(x) = h'(t)(2\pi + \frac{\partial}{\partial t}g(x))$. By definitions of h and g, and by (6), $A(x)$, $B(x)$ and $C(x) > 0$.

Let $p = (t_n, \theta_{n_j}) \in \mathrm{Per}(f)$ and ℓ be the period of p. Then $|A(p)| = |h'(t_n)| \neq 1$ and $|B(p)| = |\frac{d}{d\theta}g_{k(n)}(\theta_{n_j})| \neq 1$ by definitions of h and $g_{k(n)}$. Since

$$(7)\quad Df^{\ell}(p) = \begin{bmatrix} (A(p))^{\ell}, & 0 \\ (*), & (B(p))^{\ell} \end{bmatrix},$$

where $(*) > 0$, the eigenvalues of $Df^{\ell}(p)$, $A(p))^{\ell}$ and $(B(p))^{\ell}$ are ones of which absolute values are not equal to 1. Hence p is hyperbolic.

We finally show that for p, $q \in Per(f)$, $W^u(p)$ intersects $W^s(q)$ transversely. It suffices to show in the case of p and q are saddle points. Hence we may assume that $p \in \{t_n\} \times S^1$ and $q \in \{t_{n+1}\} \times S^1$ for some even n (if p and q are saddle points contained in the same invariant circle, then $W^u(p)$ and $W^s(q)$ hase no intersection point). Then $A(p) > 1$ and $A(q) < 1$. Let ℓ and ℓ' be periods of p and q respectively. Since $T_p(W^u(p))$ is the eigenspace corresponding to the eigenvalue $(A(p))^\ell$ of $Df^\ell(p)$, the slope of $v = (v_t, v_\theta) \in T_p(W^u(p))$, $v_\theta/v_t = (*)/[(A(p))^\ell - (B(p))^\ell] > 0$ from (7). By the same argument, for $w = (w_t, w_\theta) \in T_q(W^s(q))$, $w_\theta/w_t < 0$. If $W^u(p)$ and $W^s(q)$ have a nontransversal intersection point x, then for any $v' = (v'_t, v'_\theta) \in T_x(W^u(p))$, the slope of $Df^{m\cdot a}(x)(v') \to w_\theta/w_t$ as $m \to \infty$ where $a = \ell \cdot \ell'$.

On the other hand, since

$$Df^{m\cdot a}(x) = Df(x_{ma-1})\cdot\dots\cdot Df(x) = \begin{bmatrix} A(x_{ma-1}) \dots A(x), & 0 \\ (**) & , B(x_{ma-1})\dots B(x) \end{bmatrix}$$

where $x_i = f^i(x)$ and $(**) > 0$, the slope of $Df^{ma}(x)(v') = [B(x_{ma-1})\dots B(x)/A(x_{ma-1})\dots A(x)]\cdot v'_\theta/v'_t + (**)$. Taking x sufficiently near to p, $v'_\theta/v'_t > 0$. hence the slope of $Df^{ma}(x)(v') \to \infty$ as $m \to \infty$ since $A(x_i) \to A(q) < 1$ and $B(x_i) \to B(q) > 1$. Therefore $W^u(p)$ intersects $W^s(q)$ transversely.

For f, $\bigcup_{p \in Per(f)} W^s(p)$ is not dense in $W^s(\Omega(f))$ since $(\sqrt{2}, 2) \times S^1 \subset W^s(\{\sqrt{2}\} \times S^1)$ by (3) and $\{\sqrt{2}\} \times S^1 \subset \Omega(f) - Per(f)$.

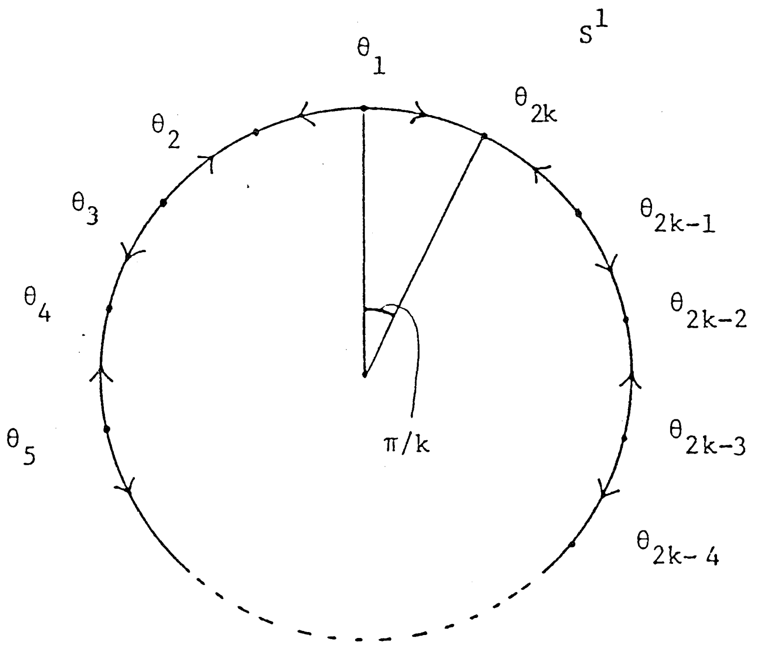
S^1
θ_1
θ_{2k}
θ_2
θ_3
θ_{2k-1}
θ_4
θ_{2k-2}
θ_5
π/k
θ_{2k-3}
θ_{2k-4}

FIGURE 1